



TITLE:

# NOTE ON P-VALENTLY BAZILEVIC FUNCTIONS

AUTHOR(S):

OWA, SHIGEYOSHI

---

CITATION:

OWA, SHIGEYOSHI. NOTE ON P-VALENTLY BAZILEVIC FUNCTIONS. 数理解析研究所講究録 1988, 664: 74-78

ISSUE DATE:

1988-07

URL:

<http://hdl.handle.net/2433/100635>

RIGHT:

# NOTES ON P-VALENTLY BAZILEVIĆ FUNCTIONS

SHIGEYOSHI OWA (近畿大・理工 尾和重義)

## ABSTRACT

The object of the present paper is to improve the former results for  $p$ -valently Bazilević functions which were recently proved by the author and others.

## 1. INTRODUCTION

Let  $A_p$  denote the class of functions of the form

$$(1.1) \quad f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\})$$

which are analytic in the unit disk  $E = \{z: |z| < 1\}$ . A function  $f(z)$  belonging to  $A_p$  is said to be  $p$ -valently starlike if and only if it satisfies the condition

$$(1.2) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in E).$$

We denote by  $S_p^*$  the subclass of  $A_p$  consisting of functions which are  $p$ -valently starlike in  $E$ .

A function  $f(z)$  belonging to the class  $A_p$  is said to be  $p$ -valently Bazilević of type  $\beta$  and order  $\gamma$  if there exists a function  $g(z)$  belonging to  $S_p^*$  such that

$$(1.3) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)^{1-\beta} g(z)^\beta} \right\} > \gamma \quad (z \in E)$$

for some  $\beta$  ( $\beta > 0$ ) and  $\gamma$  ( $0 \leq \gamma < p$ ).

---

1980 Mathematics Subject Classification (1985 Revision). Primary 30C45.

\* This research of the author was carried out at Research Institute for Mathematical Sciences, Kyoto University while the author was visiting from Kinki University.

Also we denote by  $B_p(\beta, \gamma)$  the subclass of  $A_p$  consisting of all  $p$ -valently Bazilevič functions of type  $\beta$  and order  $\gamma$  in  $E$ . The concept of Bazilevič functions was first introduced by Bazilevič [6]. Thomas [8] has called a function in the class  $B_1(\beta, 0)$  a Bazilevič function of type  $\beta$ . Further, Nunokawa [3] has proved that a function  $f(z)$  in the class  $B_p(\beta, 0)$  is  $p$ -valent in the unit disk  $E$ .

In particular, the class  $B_p(\beta, \gamma)$  for  $g(z) = f(z)$  is called is the class of  $p$ -valently starlike functions of order  $\gamma$ . Further, we note that the class  $B_p(\beta, 0)$  for  $g(z) = f(z)$  is equivalent to  $S_p^*$ .

Let  $A_p(\alpha, \beta)$  be the subclass of  $A_p$  consisting of functions which satisfy the condition

$$(1.4) \quad \operatorname{Re} \left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \beta \quad (z \in E)$$

for some real  $\alpha$  and  $\beta$ .

The class  $A_1(\alpha, 0)$  when  $p = 1$  and  $\beta = 0$  was introduced by Mocanu [2], and was studied by Miller, Mocanu and Reade [1], and Sakaguchi and Fukui [7].

Let  $C_p(\alpha, \beta)$  be the subclass of  $A_p$  consisting of functions satisfying the condition

$$(1.5) \quad \operatorname{Re} \left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} < \beta \quad (z \in E)$$

for some real  $\alpha$  and  $\beta$  ( $\beta > p$ ).

The class  $C_p(\alpha, \beta)$  was recently introduced by Nunokawa and Owa [4].

## 2. SOME PROPERTIES

In order to derive our results, we need the following lemmas.

LEMMA I ([5]). If a function  $f(z)$  belongs to the class  $A_p(\alpha, \beta)$  with  $\alpha > 0$  and  $0 \leq -\beta/\alpha \leq 1/2$ , then  $f(z) \in B_p(1/\alpha, 2^{2\beta/\alpha})$ , and therefore  $f(z)$  is  $p$ -valent in the unit disk  $E$ .

LEMMA 2 ([4]). Let a function  $f(z)$  belong to the class  $C_p(\alpha, \beta)$  with  $\alpha \neq 0$ ,  $\beta > p$ , and  $|\beta/\alpha| \leq 1/2$ . Then  $f(z)$  is  $p$ -valent in the unit disk  $E$ . Moreover, if  $0 \leq -\beta/\alpha \leq 1/2$ , then  $f(z) \in B_p(1/\alpha, 2^{2\beta/\alpha})$ .

Applying the above lemmas, we prove

THEOREM I. If a function  $f(z)$  belongs to the class  $A_p(\alpha, \beta)$  with  $\alpha > 0$  and  $0 \leq -\beta/\alpha \leq 1/2$ , then  $f(z) \in B_p(1/\alpha, p2^{2\beta/\alpha})$ .

PROOF. For a function  $f(z)$  in the class  $A_p$ , we define the function  $g(z)$  by

$$(2.1) \quad \begin{aligned} g(z) &= f(z)^{1/p} \\ &= z + g_2 z^2 + g_3 z^3 + \dots \end{aligned}$$

Then  $g(z)$  is in the class  $A_1$ , and satisfies

$$(2.2) \quad \frac{zf'(z)}{f(z)} = p \frac{zg'(z)}{g(z)}$$

and

$$(2.3) \quad 1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{zg''(z)}{g'(z)} + (p-1) \frac{zg'(z)}{g(z)}.$$

It follows from (2.2) and (2.3) that

$$(2.4) \quad \begin{aligned} (1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \\ = (p-\alpha) \frac{zg'(z)}{g(z)} + \alpha \left( 1 + \frac{zg''(z)}{g'(z)} \right). \end{aligned}$$

Therefore, we have

$$(2.5) \quad f(z) \in A_p(\alpha, \beta) \iff \operatorname{Re} \left\{ \left( 1 - \frac{\alpha}{p} \right) \frac{zg'(z)}{g(z)} + \frac{\alpha}{p} \left( 1 + \frac{zg''(z)}{g'(z)} \right) \right\} > \frac{\beta}{p}$$

$$\Longleftrightarrow g(z) \in A_1(\alpha/p, \beta/p).$$

Applying Lemma 1 for  $p = 1$ , we see that

$$g(z) \in A_1(\alpha/p, \beta/p) \implies g(z) \in B_1(p/\alpha, 2^{2\beta/\alpha}).$$

It follows that

$$f(z) \in A_p(\alpha, \beta) \implies g(z) \in B_1(p/\alpha, 2^{2\beta/\alpha})$$

$$\Longleftrightarrow \operatorname{Re} \left\{ \frac{zg'(z)}{g(z)^{1-p/\alpha} h(z)^{p/\alpha}} \right\} > 2^{2\beta/\alpha} \quad (h(z) \in S_1^*)$$

$$\Longleftrightarrow \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)^{1-1/\alpha} (h(z)^p)^{1/\alpha}} \right\} > p 2^{2\beta/\alpha} \quad (h(z)^p \in S_p^*)$$

$$\Longleftrightarrow f(z) \in B_p(1/\alpha, p 2^{2\beta/\alpha}).$$

This completes the assertion of Theorem 1.

REMARK I. Noting  $p 2^{2\beta/\alpha} \geq 2^{2\beta/\alpha}$ , we see that

$$B_p(1/\alpha, p 2^{2\beta/\alpha}) \subseteq B_1(1/\alpha, 2^{2\beta/\alpha}).$$

Thus Theorem 1 is the improvement of Lemma 1 by Nunokawa, Owa, Saitoh, Yaguchi and Lee [5].

Taking  $p = 1$  in Theorem 1, we have

COROLLARY I. If  $f(z) \in A_1(\alpha, \beta)$  with  $\alpha > 0$  and  $0 \leq -\beta/\alpha \leq 1/2$ , then  $f(z) \in B_1(1/\alpha, 2^{2\beta/\alpha})$ .

Using the same manner as in the proof of Theorem 1, we have

THEOREM 2. If a function  $f(z)$  belongs to the class  $C_p(\alpha, \beta)$  with

$\alpha \neq 0$ ,  $\beta > p$ , and  $0 \leq -\beta/\alpha \leq 1/2$ , then  $f(z) \in B_p(1/\alpha, p2^{2\beta/\alpha})$ .

Finally, letting  $p = 1$  in Theorem 2, we have

COROLLARY 2. If  $f(z) \in C_1(\alpha, \beta)$  with  $\alpha \neq 0$ ,  $\beta > 1$ , and  $0 \leq -\beta/\alpha \leq 1/2$ , then  $f(z) \in B_1(1/\alpha, 2^{2\beta/\alpha})$ .

REMARK 2. We note that Theorem 2 is the improvement of Lemma 2 due to Nunokawa and Owa [4].

## REFERENCES

- [1] S. S. Miller, P. T. Mocanu and M. O. Reade, All  $\alpha$ -convex functions are univalent and starlike, Proc. Amer. Math. Soc. 37(1973), 553 - 554.
- [2] P. T. Mocanu, Une propriété de convexité généralisée dans la théorie de la représentation conforme, Mathematica (Cluj) 11(1969), 127 - 133.
- [3] M. Nunokawa, On the Bazilevič analytic functions, Sci. Rep. Fac. Edc. Gunma Univ. 21(1972), 9 - 13.
- [4] M. Nunokawa and S. Owa, A remark on certain multivalent functions, Chinese J. Math., to appear.
- [5] M. Nunokawa, S. Owa, H. Saitoh, T. Yaguchi and S. K. Lee, On certain subclass of analytic functions. II, Sci. Rep. Fac. Edc. Gunma Univ. 36(1988), in press.
- [6] E. Bazilevič, On a class of integrability in quadratures of the Loewner-Kufarev equation, Mat. Sb. 37(1955), 471 - 476.
- [7] K. Sakaguchi and S. Fukui, On alpha-starlike functions and related functions, Bull. Nara Univ. Educ. 28(1979), 5 - 12.
- [8] D. K. Thomas, On Bazilevič functions, Trans. Amer. Math. Soc. 132(1968), 353 - 361.

Department of Mathematics  
Kinki University  
Higashi-Osaka, Osaka 577  
Japan